Physics Internal Assessment

What Is the Relationship Between Length of Resonant Coupled Oscillators and Energy Exchange Time

Word Count: 2721

Joe Liang

1 Introduction

A simple harmonic oscillator is a system in which a restoring force is directly proportional to the displacement from equilibrium. The dynamics of such a system are governed by the second-order differential equation:

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

where $x(t)$ is the displacement at time t and ω is the angular frequency of oscillation, given by:

$$\omega_0 = \sqrt{\frac{g}{L}}$$

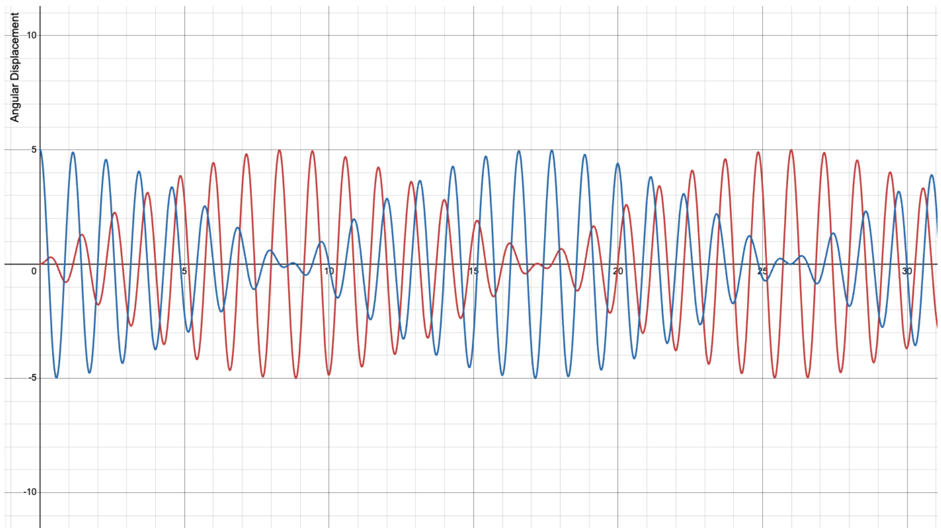
where L is the length of the pendulum and g is the gravitational constant, approximated at $9.81ms^{-2}$. The equations rely on the assumption of small angle approximations where $\sin\theta = \theta$. For this experiment, we set the maximum approximation to be $15°$.

1.1 Coupled Oscillators

When two oscillators are connected to allow motion energy to be exchanged between them, they form a system known as a coupled oscillator. In this experiment, the motion of one oscillator affects the other by oscillating the string both pendulums are connected. Refer to Figure 2a in the Methodology section to visualize the apparatus.

If the two oscillators are equal in length, they are said to be in resonance. When they are in resonance, the oscillation frequencies for the two pendulums are the same, and energy can be transferred between them efficiently. The exchange is called an energy exchange. Figure 1 demonstrates the motion of the coupled oscillators, where the blue line is the leading oscillator, and the red line is the lagging oscillator. We see the motion starts with maximum oscillations in the blue trajectory. Slowly, the energy transfers to the red trajectory, and while the blue trajectory diminishes, the red trajectory increases in amplitude. The cycle, or energy exchange, repeats continuously. In our study, the time of energy exchange will be defined as the time difference from the peak of the blue oscillation, to the first minimum of oscillation. We will derive the mathematical background of motion after introducing normal modes.

Figure 1: Angular displacements of a coupled oscillator.



1.2 Normal Modes

The interaction of the coupled oscillators gives rise to normal modes. Normal modes are independent oscillation patterns in which a system oscillates at the same frequency. In our study, the system exhibits two distinct normal modes: swinging together and swinging opposite. Each mode corresponds to a unique frequency. When the two pendulums oscillate in opposite directions, the pivot point of both pendulums is on the string. In the other normal mode, when they are oscillating in the same direction, the entire system oscillates, and the pivot point will be higher, at the point of connection between the string and the beam. The result is two different normal frequencies:

$$\omega_1 = \sqrt{\frac{g}{L}}, \quad \omega_2 = \sqrt{\frac{g}{L+a}} \tag{1}$$

where a is the vertical length difference from the first pivot point to the second.

Importantly, the equation of motion for the system is a linear combination of the normal modes [Morse, 1936], expressed as:

$$x(t) = A\cos(\omega_1 t + \phi_1) + B\cos(\omega_2 t + \phi_2)$$

where A, B, ϕ_1, and ϕ_2 are initial conditions. To analyze the exchange of energy between the oscillators, we simplify the model by assuming that $A = B$ (due to symmetry) and $\phi_1 = \phi_2$.

Using the sum to product sinusoidal identities, we further simplify the equation to:

$$x(t) = 2A\cos\left(\frac{\omega_1 + \omega_2}{2}t\right)\cos\left(\frac{\omega_1 - \omega_2}{2}t\right)$$

The term $\cos\left(\frac{\omega_1 - \omega_2}{2}t\right)$ has a significantly longer period since the term inside the sinusoidal is the subtraction of terms. Hence, it models the amplitude of the entire function. The period of the oscillations can then be expressed as:

$$T_{\text{period}} = \frac{2\pi}{\frac{\omega_1 - \omega_2}{2}} = \frac{4\pi}{\omega_1 - \omega_2}$$

Given that the frequencies are derived from the lengths of the pendulums, the time of energy exchange is a function of both pendulum lengths. Substituting the frequencies with lengths from Equation 1, the relationship between the energy exchange period and the pendulum lengths is given by:

$$T_{\text{period}} = \frac{4\pi}{\sqrt{\frac{g}{L}} - \sqrt{\frac{g}{L+a}}}$$

However, the observed period is also influenced by a coupling coefficient [Olsen, 1945]. The coupling coefficient accounts for non-ideal factors, such as variations in coupling strength due to separation distance, frictional effects, or minor differences in pendulum characteristics. Additionally, the time of energy exchange is only a fraction of the period. The many factors combined make it difficult to estimate the coupling coefficient. However, since the coupling coefficient only modifies the equation linearly, we can still obtain a concrete equation of proportionality:

$$T_{\text{exchange}} \propto \frac{1}{\sqrt{\frac{g}{L}} - \sqrt{\frac{g}{L+a}}} \tag{2}$$

In this investigation, we aim to explore how the relative lengths of two resonant harmonic oscillators affect the time of energy exchange between them, and how these variations influence the system's behaviour over time. However, while Equation 2 models the relationship, it does not allow for linear analysis.

1.3 Taylor Series Expansion for Linearization

Instead, to further simplify this relationship, we can perform a Taylor series expansion on the expression to obtain a polynomial form, assuming $a < L$. A final equation in polynomial form allows for linear analysis through taking L to the obtained degree.

A Taylor series expansion is an infinite sum of terms that are expressed in terms of the function's derivatives. Terms of higher-order derivatives attribute minimally to the accuracy of the approximations, so obtaining the first two terms is sufficient for our case. Using a first-order Taylor expansion for the square root terms, we have:

$$\sqrt{\frac{g}{L+a}} \approx \sqrt{\frac{g}{L}} - \frac{a}{2} \cdot \frac{g}{L^{3/2}}$$

Substituting this into the denominator, we get:

$$\sqrt{\frac{g}{L}} - \sqrt{\frac{g}{L+a}} \approx \frac{a \cdot g}{2L^{3/2}}$$

Thus, the expression for T_{exchange} becomes:

$$T_{\text{exchange}} \propto \frac{\pi}{\frac{a \cdot g}{2L^{3/2}}}$$

Or more simply:

$$T_{\text{exchange}} \propto L^{3/2} \tag{3}$$

2 Variables and Measurement

Independent Variable: Length of the Pendulum

Measurement: Length measured from the point of suspension to the mass using a measuring tape. The error of the instrument is ± 0.005 m.

Range: The lengths will vary from 0.200 m to 1.500 m, with increments of 0.100 m. The minimum range is present due to the differences in the two normal modes of the system. When the length of the pendulums is too small, the normal modes have a frequency difference that is too high to align with the model. The 1.500 m maximum is sufficient to observe the relationship. Increments of 0.100 m allow for a balance between obtaining precise data points and redundancy.

Dependent Variable: Time of Energy Exchange

Measurement: The time for the leading pendulum to transfer energy to the lagging pendulum and return to the starting position. Referring to figure 1, the energy exchange is complete when the angular velocity is at its lowest in the trough of its motion. From our small angle approximation, the angular velocity is similar to the horizontal velocity. In tracking software, find the time between the release of the pendulum and the time of the lowest horizontal velocity to obtain the time of energy exchange.

Error & Range: The error associated with the measurement has two parts. The negligible component is the absolute error from the instrument. The frame rates of 60 FPS recording results in an error of at most $\pm$ $0.02s$. The more significant error results from the difficulty in judging the time of release and minimum angular velocity, since the value will be made with a human observation. To account for the human error, a total of 15 trials will be conducted at each interval of length, allowing for calculations of standard deviations to quantify the error.

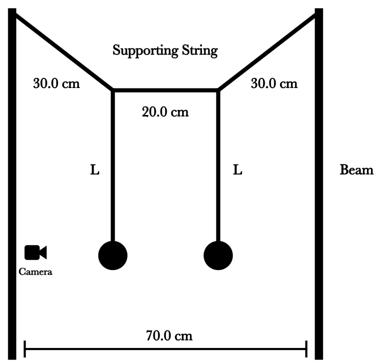
Controlled Variables

Amplitude: Simple harmonic motion is only applicable for drop angles under $\approx 15°$. We set the drop angle to the maximum of the assumption since higher drop angles have angular displacements that are more pronounced and more easily analysed. The angle must be the same since low drop angles typically lead to weaker coupling.

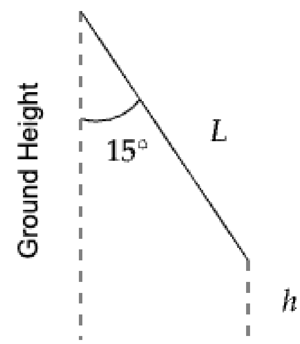
Mass of the Bob: Unlike typical pendulums, the coupled pendulum system is affected by the mass of the bob. Heavier masses have stronger and unpredictable effects on the coupling between the pendulum masses, in the form of unpredictable vibrations of the strings. In addition, a mass that is too light may be affected by air resistance. From preliminary testing, a pendulum bob mass of 20.00 g minimizes both effects.

Separation of Pendulums: The separation between pendulums determines the coupling strength. The closer the separation, the stronger the coupling and the faster the energy exchange. Preliminary testing shows a 0.200 ± 0.005 m separation between the pendulums resulting in a stable system. We will use this value for the experiment.

3 Methodology



(a) Apparatus



(b) Height above ground propagation. The formula is h = ground height$-L\cos(15°)$

Figure 2: Methodology schematics.

1. Set up apparatus as in Figure 2a. Place a camera perpendicular to the motion of the pendulums.

2. Tie pendulums using strings of equal length onto the string across the beams. Start with a string length of 0.20 m. Tie a 20.00 g spherical steel mass at the end of each string.

3. Displace the pendulum by 15 degrees. Use the height propagation outlined in Figure 2b to obtain the height of mass for a 15° drop.

4. Release the pendulum. The energy will begin to exchange, and the pendulum released will oscillate slower while the other will oscillate faster. Track the leading pendulum's velocity with tracking software.

5. When the horizontal velocity of the leading pendulum is at its lowest, the energy is fully exchanged. The time between the release and the lowest horizontal velocity is the time exchanged.

6. Repeat trials with the same lengths for an additional 14 times to obtain 15 trials for the specific length.

7. Continue to increase the pendulum length by 0.100 m and repeat the data collection until the pendulum lengths ranging from 0.200 m to 0.150 m have been completed.

4 Data Analysis

4.1 Single trial

Figure 3: Horizontal displacement of the leading pendulum over time.

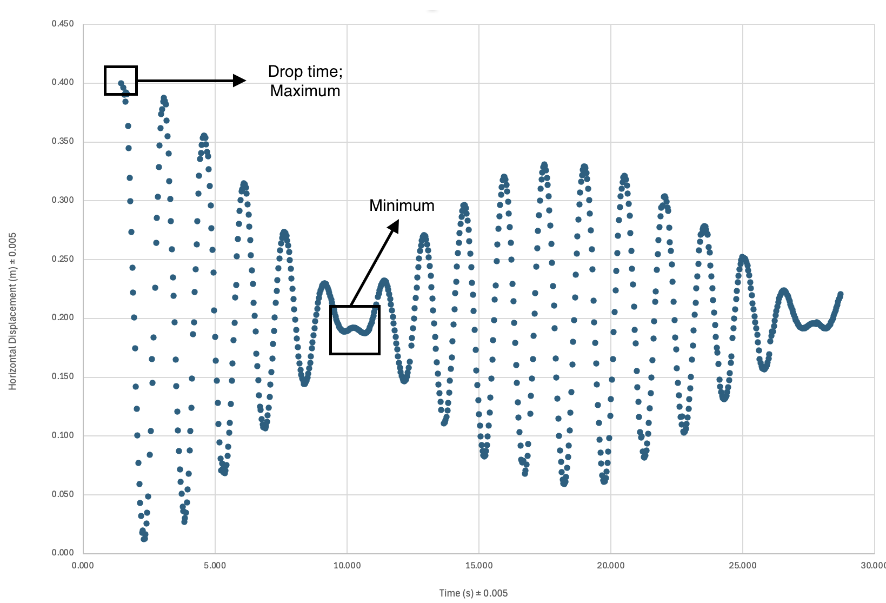


Figure 3 shows the horizontal displacement over time of a trial with a length of 0.500 m. To find the time of energy exchange, subtract the time of the lowest horizontal velocity from the moment of release. Finding the lowest horizontal velocity is the most difficult part of the process. We first locate the oscillation with the smallest peak, labelled in Figure 3. We will set the maximum of the peak as the lowest horizontal velocity because the derivative at that point is zero. Additionally, the error will be $0.1s$ since that is the width of the peak. For this trial, we judge the peak as $10.307s$, with the release time at $1.435s$. The energy exchange is then $10.307 - 1.435 = 8.872 \approx 8.9 \pm 0.1s$. If we repeat the analysis for all trials, we obtain Table 1.

Table 1: Quantitative data on the relationship between oscillator length and time of energy exchange.

Length (m) ± 0.005	**Time (s)** $\pm 0.1s$														
	Trial 1	**Trial 2**	**Trial 3**	**Trial 4**	**Trial 5**	**Trial 6**	**Trial 7**	**Trial 8**	**Trial 9**	**Trial 10**	**Trial 11**	**Trial 12**	**Trial 13**	**Trial 14**	**Trial 15**
0.200	2.9	2.5	2.5	2.9	2.5	2.7	2.7	3.0	2.9	2.8	2.7	2.6	2.7	3.1	2.8
0.250	3.9	3.8	3.4	3.7	3.7	3.7	3.4	3.8	3.7	3.3	3.7	3.9	3.8	3.8	4.0
0.300	4.6	4.5	4.3	4.2	4.4	4.7	4.3	4.4	4.6	4.5	4.5	4.4	4.2	4.5	4.3
0.350	5.4	5.7	5.1	5.4	5.5	5.3	5.3	5.6	5.6	5.5	5.3	5.4	5.4	5.2	5.5
0.400	6.4	6.0	6.6	6.5	6.6	6.6	6.7	6.5	6.5	6.9	6.6	6.8	6.9	6.9	6.4
0.450	7.5	7.8	7.6	7.6	7.5	7.7	7.8	7.6	7.7	7.9	7.8	8.1	7.6	7.4	7.5
0.500	8.8	8.5	8.8	8.8	8.9	9.0	8.9	8.8	8.9	8.6	8.5	8.7	9.0	8.7	8.8
0.550	9.7	10.0	9.6	10.5	10.0	9.5	10.5	9.8	10.1	10.1	9.8	9.2	9.5	10.2	10.0
0.600	10.9	10.9	11.2	11.3	11.2	11.0	11.2	11.0	10.6	10.9	10.7	11.3	11.0	11.2	11.0
0.650	12.0	12.0	12.4	12.5	12.1	11.8	12.0	12.3	12.5	11.7	12.6	12.1	12.0	12.4	12.9
0.700	13.1	14.0	13.4	12.9	13.4	13.4	13.0	12.8	13.4	13.5	13.4	13.7	13.2	13.3	13.2
0.750	14.8	14.5	14.9	15.1	14.8	14.4	14.8	14.4	14.6	15.1	14.6	14.3	14.6	14.4	14.6
0.800	16.3	16.5	16.4	16.1	16.2	16.3	16.2	16.2	16.4	16.2	16.2	16.3	16.3	16.3	16.4
0.850	17.6	17.6	18.1	18.1	17.9	17.9	17.6	17.6	18.0	17.8	17.7	18.1	17.6	17.5	17.8
0.900	19.4	18.8	18.9	19.0	18.5	18.8	19.4	19.2	19.0	19.0	19.2	18.9	19.2	19.1	18.6
0.950	20.4	20.5	20.4	20.4	20.5	20.7	20.4	20.6	20.4	20.5	20.5	20.5	20.6	20.4	20.5
1.000	22.8	22.8	21.6	21.0	21.6	23.2	21.8	21.8	22.0	22.2	22.4	23.0	22.2	22.9	22.9
1.050	23.4	23.5	24.0	23.3	24.2	24.4	23.6	23.2	24.1	23.8	23.5	24.0	23.5	23.9	23.2
1.100	25.7	25.5	25.3	25.2	25.2	25.5	25.4	25.4	25.2	25.2	25.7	25.8	25.6	25.5	25.6
1.150	26.8	26.8	26.8	26.7	26.8	26.9	27.2	26.7	26.6	26.8	26.9	26.9	27.1	26.9	27.0
1.200	28.7	28.9	28.1	28.5	28.1	29.0	28.5	29.0	29.0	28.6	28.4	28.8	29.0	28.5	28.5
1.250	30.8	30.3	30.2	30.8	30.6	30.6	30.6	30.9	30.5	30.4	30.7	30.6	30.3	30.7	30.6
1.300	31.8	31.9	31.9	31.9	31.4	31.5	32.3	31.6	31.6	32.4	31.9	31.9	32.0	31.9	31.5
1.350	33.5	33.7	33.4	34.4	33.1	34.0	33.8	34.6	33.8	33.3	34.0	33.6	33.8	33.3	33.1
1.400	35.9	35.8	35.8	35.4	35.6	35.2	35.3	35.5	35.5	35.4	35.8	36.1	35.9	35.6	35.4
1.450	37.3	36.8	37.4	37.0	37.4	37.3	37.1	37.8	37.5	37.8	37.3	37.7	37.2	37.7	37.5
1.500	39.7	39.0	38.8	39.2	38.9	39.2	39.2	39.5	39.1	38.8	39.0	39.3	39.1	39.5	39.2

Table 2: Qualitative Observations

Observations
Upon the start of each trial, when we pull on the leading pendulum, the system vibrates slightly, causing the lagging pendulum to oscillate slowly.
When the string length of the pendulum exceeded $\approx$ 1.00 m, it would get tangled in itself, spinning vigorously during the trial. These trials are disregarded but should be something to be aware of.

4.2 Numerical analysis

To analyze the raw data in Table 1, we calculate the mean and standard deviation for each set of trials at each pendulum length. The standard deviation, SD, is calculated using:

$$SD = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{x})^2}$$

where $\bar{x}$ is the mean, x_i represents each individual measurement, and N is the total number of trials.

Sample Calculation for Length 0.200 m

For the length $L = 0.200\,\text{m}$ we obtain the mean of $2.7s$. The standard deviation is then

$$SD = \sqrt{\frac{1}{15-1}\sum_{i=1}^{15}(x_i - 2.7)^2}$$
$$SD = 0.2\,\text{s}$$

Two standard deviations will capture 95.4% of the data. Thus, for $L = 0.200\,\text{m}$, the mean time of energy exchange is 2.7 s with an error of $2 \times 0.2 = 0.4\,\text{s}$. By repeating these calculations for each length, we obtain Table 3.

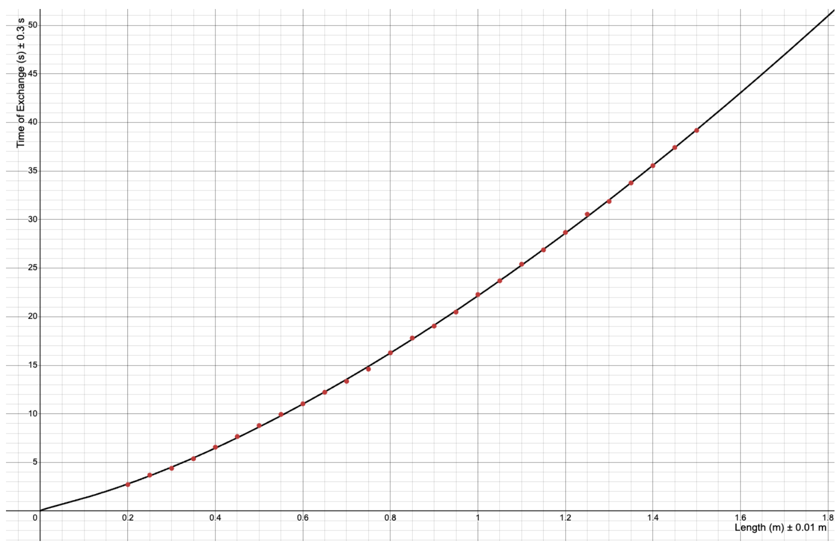
4.3 Higher degree analysis

Table 3: Quantitative data of the length of oscillators and time of energy exchange with corresponding standard deviations.

Length ± 0.005 **(m)**	**Time (s)** $\pm 2SD$	**2 SD (s)**	**Length** ± 0.005 **(m)**	**Time (s)** $\pm 2SD$	**2 SD (s)**
0.200	2.7	0.3	0.900	19.0	0.5
0.250	3.7	0.3	0.950	20.5	0.2
0.300	4.4	0.3	1.000	22.3	1.2
0.350	5.4	0.4	1.050	23.7	0.8
0.400	6.6	0.2	1.100	25.4	0.4
0.450	7.7	0.2	1.150	26.9	0.4
0.500	8.8	0.3	1.200	28.7	0.6
0.550	10.0	0.7	1.250	30.6	0.4
0.600	11.0	0.4	1.300	31.9	0.6
0.650	12.2	0.6	1.350	33.8	0.8
0.700	13.4	0.6	1.400	35.6	0.9
0.750	14.6	0.6	1.450	37.4	0.9
0.800	16.3	0.3	1.500	39.2	0.8
0.850	17.8	0.5			

We plot the first unmodified graph on Desmos to visualize the trend and utilize the computational software to obtain values of proportionality. Note, we are saving the error analysis for the linearized graphs, and the polynomial graph here is simply for the purpose of obtaining the variable a to confirm the accuracy of the experiment. Additionally, the relative error is negligible. For the above reasons, no error bars are present on Figure 4.

Figure 4: Relationship between length of pendulums and time of energy exchange.



From the Desmos plot, we input the derived equation:

$$T = \frac{b}{\sqrt{\frac{g}{l}} - \sqrt{\frac{g}{l+a}}}$$

The computer finds the most suitable values as $b = 5.029$ and $a = 0.162$. Substituting, with units, we get

$$T = \frac{5.029}{\sqrt{\frac{9.81ms^{-2}}{l}} - \sqrt{\frac{9.81ms^{-2}}{l+0.162m}}} \tag{4}$$

The significant variable is a. From the background, we understand that a is what offsets the two normal modes. The offset is the difference in length between the longer and shorter oscillation points. From the diagrams of the apparatus in the Methodology section, we see the value of a in reality is 0.166 ± 0.005 m. Compared with the computer's 0.162m, the difference between the theoretical and the reality is minimal. It is within the absolute error of the instrument of ± 0.005 m. The accuracy reflects the accuracy of the model.

4.4 Linear analysis

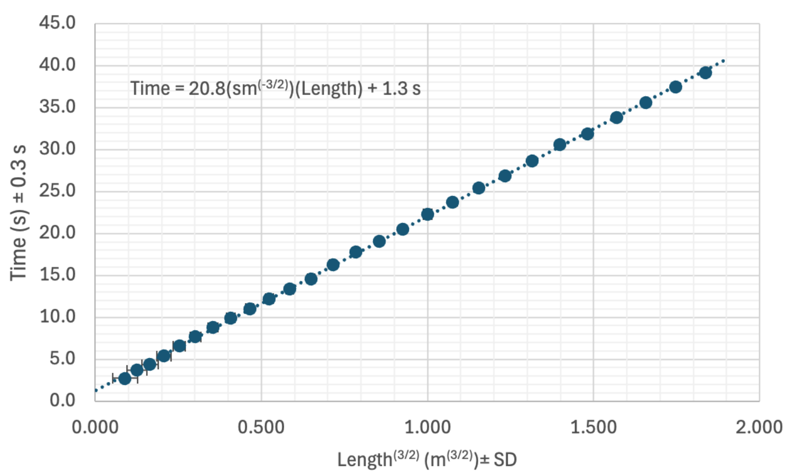
Next, we can linearize the data points to facilitate linear analysis. Recall from the background that the proportionality equation is given by $T \propto L^{3/2}$, from the Taylor series expansion. By transforming our data points for L by raising each to the power of $\frac{3}{2}$, we obtain the values presented in Table 1. With this transformation, the data now follows a linear trend, making it suitable for linear analysis.

To determine the error in our transformed data, let ϵ_L be the percentage error in L. Since L is raised to the power of $\frac{3}{2}$, the propagated error becomes:

$$\epsilon_{L^{3/2}} = \frac{3}{2} \cdot \epsilon_L$$

We obtain Table 4 for the calculations and Figure 5 for the graph. The standard deviation of the time remains the same as Table 3 since no changes have been made.

Figure 5: Linearized relationship between length of pendulums and time of energy exchange.



Note: Error bars are mostly too small to be visible.

In cases of high correlation and negligible error, a minimum-maximum gradient analysis of the trend line would not provide meaningful analysis due to its negligible difference. Instead, we calculate the r^2 value to quantify the fit. The r^2 value can be obtained using the formula:

Table 4: Linearized data between the length of pendulums and time of energy exchange.

Length$^{\frac{3}{2}}$ ($m^{\frac{3}{2}}$) $\pm\epsilon_L$	**Time (s)** $\pm 2SD$	**Length**$^{\frac{3}{2}}$ ($m^{\frac{3}{2}}$) $\pm\epsilon_L$	**Time (s)** $\pm 2SD$
0.089 ± 0.038	2.7 ± 0.3	0.854 ± 0.008	19.0 ± 0.5
0.125 ± 0.030	3.7 ± 0.3	0.926 ± 0.008	20.5 ± 0.2
0.164 ± 0.025	4.4 ± 0.3	1.000 ± 0.008	22.3 ± 1.2
0.207 ± 0.021	5.4 ± 0.4	1.076 ± 0.007	23.7 ± 0.8
0.253 ± 0.019	6.6 ± 0.2	1.154 ± 0.007	25.4 ± 0.4
0.302 ± 0.017	7.7 ± 0.2	1.233 ± 0.007	26.9 ± 0.4
0.354 ± 0.015	8.8 ± 0.3	1.315 ± 0.006	28.7 ± 0.6
0.408 ± 0.014	10.0 ± 0.7	1.398 ± 0.006	30.6 ± 0.4
0.465 ± 0.013	11.0 ± 0.4	1.482 ±0.006	31.9 ± 0.6
0.524 ± 0.012	12.2 ± 0.6	1.569 ± 0.006	33.8 ± 0.8
0.586 ± 0.011	13.4 ± 0.6	1.657± 0.005	35.6 ± 0.9
0.650 ± 0.010	14.6 ± 0.6	1.746 ± 0.005	37.4 ± 0.9
0.716 ± 0.009	16.3 ± 0.3	1.837 ± 0.005	39.2 ± 0.8
0.784 ± 0.009	17.8 ± 0.5		

$$r^2 = 1 - \frac{\sum(y_i - \hat{y}_i)^2}{\sum(y_i - \bar{y})^2}$$

where y_i are the observed values, $\hat{y}_i$ are the predicted values from the linear model and $\bar{y}$ is the mean of the observed values. The r^2 value, which ranges from 0 to 1, represents the degree of correlation between the data and the trend line. The calculated r^2 value of this experiment is 0.98. The high value signifies a strong correlation.

Despite the strong correlation, there is one major difference between our data and the model. Since the model's prediction is a relationship that is directly proportional, the y-intercept of the linearized data should cross the origin. Instead, the y-intercept is $1.3s$, which is well outside the theoretical prediction, even considering errors. We address the asymmetry in evaluation, classifying the shift as a systematic error.

5 Conclusion

The data collected in this study strongly supports the hypothesis that the time of energy exchange between two coupled pendulums is influenced by their relative lengths. The plot of time of energy exchange T against pendulum length L exhibits a clear trend consistent with the derived relationship $T \propto L^{3/2}$. The strong r^2 value of 0.98 reflects an excellent fit, indicating that the model accurately captures the relationship between pendulum length and exchange time. Furthermore, the negligible standard deviation throughout the data reinforces the high precision of the measurements. The constant a, the predicted value of the difference in length, aligns closely with the real value, indicating that the length-dependent frequency differences between normal modes are well captured by the model.

Overall, the consistency between theoretical and empirical values suggests that the experimental apparatus and setup were effectively controlled and that the model is robust in predicting behaviour across the tested range of lengths.

In comparison with existing literature, coupled oscillator systems are a well-researched topic. However, the specific relationship between pendulum length and energy exchange time in resonant harmonic oscillators is not directly outlined in existing literature. Instead, the theory is largely self-derived. The findings, however, align with the general principle that an increase in pendulum length results in longer periods of oscillation and, consequently, an extended energy exchange time. Thus, this experiment contributes a unique perspective to the understanding of coupled pendulum dynamics, grounded in consistent empirical evidence and agreement on broader mechanical principles observed in coupled oscillatory systems.

To conclude, the data provides strong support for the derived relationship and, given the low uncertainty and high accuracy of the measurements, confidently validates the model proposed in this experiment. Future research could extend this approach by examining the consistency of the trend with longer oscillator lengths or by exploring other coupling factors or configurations, thereby further enriching our understanding of energy exchange dynamics in resonant systems.

6 Evaluation

Nonetheless, several factors may have introduced sources of error that influenced the accuracy of the measurements.

Firstly, there is a noticeable loss of energy over time, which is evident in the decrease in amplitude from one maximum to the next. We can see the diminishing heights between the periods in Figure 3. The first maximum lies on a horizontal displacement of $\approx 0.400m$, while the second one lies on $\approx 0.335m$. The amplitude diminishes due to air resistance and minor internal friction within the pendulum's pivot, which slowly drains energy from the system and increases the measured energy exchange time.

Additionally, controlling the pre-trial vibrations of the lagging pendulum was difficult. As outlined in the qualitative data, there are small vibrations of the system when we initially pull back the leading pendulum. Any initial disturbance increases the time for the system to achieve the phase shift for the energy exchange pattern. Consequently, there is a systematic upward shift in the energy transfer times.

As a consequence of the energy loss and uncontrolled vibrations, the energy exchange time shifted systematically upwards. According the Equation 3, the y-intercept of the linearized graph should be at zero. However, the deviations increased the intercept to $1.3s$. The discrepancy of $1.3s$ is well outside the range of the error, confirming the presence of the systematic error. Of course, taking the cubic function to linearize the variables exaggerates the displacement, but it is still significant regardless.

In future experimentation, we can change a few things to increase accuracy of the results. Firstly, smoother materials for the strings may decrease the energy loss due to friction. Secondly, a contraption or assistant to hold on to the lagging pendulum and its pivot point as the leading pendulum is held back will reduce initial vibrations. Together, the changes could reduce systematic increases in exchange time, which is the most significant weakness of this experiment.

7 Bibliography

[Morse, 1936] Morse, P. M. (1936). *Vibration and Sound*. First edition.

[Olsen, 1945] Olsen, L. O. (1945). Coupled pendulums: An advanced laboratory experiment. *American Journal of Physics*, 13(5):321–324.